

Imaging the 3-D cosmological mass distribution with weak gravitational lensing

A. N. Taylor

Institute for Astronomy, Royal Observatory, Blackford Hill, Edinburgh, EH9 3HJ, U. K.

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I show how weak gravitational lensing can be used to image the 3-D mass distribution in the Universe. An inverse relation to the lensing equation, relating the lensing potential evaluated at each source to the full 3-D Newtonian potential, is derived. I consider the normal modes of the lensing problem and clarify the equations using a small-angle approximation. Finally I consider the prospects of using this method to estimate the 3-D matter distribution from a realistic galaxy lensing survey.

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I. INTRODUCTION

Gravitational lensing provides us with the most direct and cleanest of methods for probing the distribution of matter in the universe [1,2]. The lensing effect arises from the deflection of light by perturbations in the metric. These deflections stretch and contract bundles of light rays, causing the distortion of background galaxy images. Hence gravitational lensing does not depend on any assumptions about the state of matter. These distortions manifest themselves as a shear distortion of the source galaxy image [4,5], or a change in the surface number density of source galaxies due to magnification [6–8] and can be used to map the two dimensional projected matter distribution of cosmological structure. As the matter content of the universe is dominated by non-baryonic and non-luminous matter, gravitational lensing is the most accurate method for probing the distribution of this Dark Matter. Imaging of the Dark Matter distribution is a vital key to understanding its nature.

The measurement of the gravitational mass distribution in clusters of galaxies using gravitational lensing shear and magnification effects is now a well established technique, while on larger scales the detection of a cosmic shear signal [9] shows that the cosmological matter distribution can also be probed this way. A problem with these methods is the limited use of depth information, such as spectroscopic or photometric redshifts. Usually the intervening lensing matter distribution is approximated by a sheet. Some depth information can be introduced by lens tomography [10,11], where the background source galaxies are divided into bins and the matter distribution can be approximated by a series of sheets. What is lacking is a method for estimating the full 3-D matter distribution from gravitational lensing. Here I address this problem.

II. METHOD

The metric of a perturbed Friedmann-Lemaître-Robertson-Walker universe in the conformal Newtonian,

or longitudinal, gauge is

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Phi)dr_i dr^i \quad (1)$$

where Φ is the Newtonian potential, a is the cosmological scale factor, and we have assumed a spatially flat universe for simplicity. The Newtonian potential is related to the matter density field by Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho_m \delta a^2 = \frac{3}{2} \lambda_H^{-2} \Omega_m a^{-1} \delta, \quad (2)$$

where $\lambda_H = 1/H_0 \approx 3000 h^{-1} \text{Mpc}$ is the Hubble length, and Ω_m is the present-day mass-density parameter.

The lensing potential, ϕ , for a source in a spatially flat universe at distance r is given by [2,3]

$$\phi(\mathbf{r}) = 2 \int_0^r dr' \left(\frac{r - r'}{rr'} \right) \Phi(\mathbf{r}'), \quad (3)$$

measured at an angular position $\hat{\mathbf{r}}$ on the sky. We have assumed the Born approximation, where the light path is unperturbed. This equation shows that the lensing potential is a radial projection of the 3-D gravitational potential, with a radial Greens function $G(r, r') = 2(r - r')/rr' \Theta(r - r')$, where $\Theta(r)$ is the Heaviside function.

The inverse relation to equation (3) is

$$\Phi(\mathbf{r}) = \frac{1}{2} \partial_r r^2 \partial_r \phi(\mathbf{r}) \quad (4)$$

where $\partial_r = \hat{\mathbf{r}} \cdot \nabla$ is the radial derivative, and we assume the lensing potential has been appropriately smoothed to allow differentiation. This can be verified by substitution into equation (3) and integrating by parts.

The lensing potential is not an observable. The observables are the dimensionless, symmetric, tracefree shear matrix, γ_{ij} , which describes the distortion of the lensed image and the magnification, μ , which describes the change in area. The shear metric is

$$\gamma_{ij} = \left(\partial_i \partial_j - \frac{1}{2} \delta_{ij}^K \partial^2 \right) \phi, \quad (5)$$

where $\partial_i \equiv r(\delta_{ij} - \hat{r}_i \hat{r}_j) \nabla_j = r(\nabla_i - \hat{r}_i \partial_r)$ is a dimensionless, transverse differential operator, and $\partial^2 \equiv \partial_i \partial^i$ is the transverse Laplacian. The indices $(i, j) = (1, 2)$. In this expression we have assumed a flat sky. The magnification is given by [6]

$$\mu = |(1 - \kappa)^2 - \gamma^2|^{-1} \approx 1 + 2\kappa. \quad (6)$$

The second approximation holds for weak magnification, where the lens convergence, κ , is defined by the 2-D Poisson's equation,

$$\kappa = \frac{1}{2} \partial^2 \phi. \quad (7)$$

In principle there is another second-rank tensor which can be formed from a scalar potential;

$$B^{ij} = \varepsilon_{(m}^i \partial_n) \partial^j \phi_B, \quad (8)$$

where ε_j^i is the 2-D antisymmetric Levi-Civita tensor, the brackets indicate symmetrization of the indices and ϕ_B is a pseudo-scalar potential. The symmetric tensor B , and pseudo-scalar ϕ_B have odd parity and therefore cannot correspond to the parity invariant matter density field. Hence we expect $B = 0 = \phi_B$, and can use this to investigate noise and boundary effects in lensing.

The lensing potential can be estimated from the shear field by the generalised Kaiser-Squires [5] relation;

$$\tilde{\phi} = 2\partial^{-4} \partial_i \partial_j \gamma_{ij}, \quad (9)$$

where ∂^{-2} is the inverse 2-D Laplacian operator. In practice the shear field is only discretely sampled by galaxies so we must smooth the shear field to perform the differentiation. This also serves to make the uncertainty on the measured shear field finite, since each source has an unknown intrinsic ellipticity. This smoothing need only be perpendicular to the light path. There may also be intrinsic alignments of galaxies due to tidal effects during their formation [12,13]. However, these appear to be small at large distances [14].

The observable shear and convergence allow us to measure the lensing potential up to an arbitrary function of r , the radial distance;

$$\tilde{\phi}(\mathbf{r}) = \phi(\mathbf{r}) + \psi(r), \quad (10)$$

where $\tilde{\phi}$ is the measured gravitational potential and $\psi(r)$ is a solution to the equations

$$\left(\partial_i \partial_j - \frac{1}{2} \delta_{ij}^K \partial^2 \right) \psi = 0. \quad (11)$$

This sheet-like gauge freedom arises because at each distance the shear and convergence define the potential only up to an arbitrary constant. In principle this can arbitrarily change as a function of distance. As the reconstruction of the Newtonian potential requires radial derivatives we have to smooth in the radial direction.

This also transforms ψ from an arbitrary radial function to one which is smooth on the scale of the radial smoothing radius. The radial gauge freedom we see here is related to the so-called sheet-mass degeneracy which also arises as a constant of integration when deriving the convergence from the shear [15]. However it is important to note that the radial freedom we see here also arises if the convergence is used to estimate the lensing potential.

Since we expect the true lensing potential field to respect statistical isotropy and homogeneity, the observed $\tilde{\phi}$ -field must obey the equation

$$\langle \partial_r \tilde{\phi} \rangle = \frac{\partial}{\partial r} \psi, \quad (12)$$

where the angled brackets $\langle \dots \rangle$ denote ensemble averaging, or tangential averages over light-paths at the same distance. Substituting back into equation (4) we find

$$\Phi = \frac{1}{2} \partial_r r^2 (\partial_r \tilde{\phi} - \langle \partial_r \tilde{\phi} \rangle) \quad (13)$$

is an unbiased estimate of the Newtonian gravitational potential. The derivation of equation (13) is the main result of this paper, and demonstrates that the full 3-D Newtonian potential, and hence the matter-density field, can be reconstructed from weak lensing observations.

In practice ψ will not be a major problem for large surveys, since the boundary conditions used for the inversion can be used to set the mean potential at a given radius to zero. This is fine if the survey is large enough and the mean potential is zero, but if the angular size of the survey is small the potential may not average to zero.

A. Normal modes

The lensing functions can conveniently be expanded in normal modes, this time taking the curvature of the sky into account. We define the 3-D spherical harmonic modes of a field by

$$\phi_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \int_0^{r_\infty} dr r^2 \int_{4\pi} d\Omega \phi(\mathbf{r}) j_\ell(kr) Y_{\ell m}^*(\Omega), \quad (14)$$

where $r_\infty = \int_0^\infty dt/a(t)$ is the causal horizon. For a spatially open universe with hyperbolic geometry the usual spherical Bessel functions can be generalised to the hyper-spherical Bessel functions, $j_\ell(x) \rightarrow X_\ell(\Omega_K, x)$.

The harmonic moments of the potential are related to the convergence by a 2-D Poisson's equation, yielding [3]

$$\kappa_{\ell m}(k) = -\frac{1}{2} \ell(\ell+1) \phi_{\ell m}(k). \quad (15)$$

Similarly the shear field can be decomposed into tensor spherical harmonics (e.g. [16]),

$$\gamma_{ij}(\hat{\mathbf{r}}) = \sqrt{\frac{2}{\pi}} \int_0^\infty k^2 dk \sum_{\ell m} \gamma_{\ell m}(k) j_\ell(kr) Y_{(\ell m)ij}^E(\hat{\mathbf{r}}), \quad (16)$$

where

$$Y_{(\ell m)ij}^E(\hat{\mathbf{r}}) = \sqrt{\frac{2(\ell-2)!}{(\ell+2)!}} (Y_{(\ell m);ij} - 1/2 g_{ij} Y_{(\ell m);c}^c) \quad (17)$$

and where g_{ij} and “;” are the metric and covariant derivative on the 2-sphere respectively. The $\gamma_{\ell m}(k)$ are then related to the lensing potential field by [3]

$$\gamma_{\ell m}(k) = \frac{1}{2} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \phi_{\ell m}(k). \quad (18)$$

Similar expressions exist for the non-gravitational parity-violating shear term.

The normal modes of the Newtonian potential are then related to the lensing potential by

$$\Phi_{\ell m}(k) = \frac{1}{2} \int dk' [\ell(\ell+1)\delta_D(k'-k) - \alpha_\ell(k, k')] \phi_{\ell m}(k') \quad (19)$$

where

$$\alpha_\ell(k, k') = \frac{\pi}{2} \int_0^\infty dr (k'r)^4 j_\ell(kr) j_\ell(k'r). \quad (20)$$

The first term of equation (19) is simply minus the lens convergence, which reflects the angular distribution of the matter field. The second term contains all of the distance information and shows that the radial modes are correlated, as lensing accumulates with distance.

B. The small-angle approximation

For small-angles these equations can be simplified. It is useful to consider plane-wave solutions where the inversion equation becomes

$$\Phi = -\frac{1}{2} r^2 k^2 \mu^2 \phi, \quad (21)$$

and now $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{r}}$ is a cosine angle. Combining this with the plane-wave solution for the convergence,

$$\kappa = -\frac{1}{2} r^2 k^2 (1 - \mu^2) \phi, \quad (22)$$

and Poisson's equation we find

$$\delta = -\frac{2}{3} k^2 \lambda_H^2 \left(\frac{a}{\Omega_m} \right) \left(\frac{\mu^2}{1 - \mu^2} \right) \kappa. \quad (23)$$

This relation can be readily understood if we consider modes parallel and perpendicular to the light-path, $k_{||} = k\mu$ and $k_{\perp} = k(1 - \mu^2)^{1/2}$ respectively. While k_{\perp} is related to the tangential scale of the structure being probed, the radial mode, $k_{||}$ arises from the integration

along the light-path and so shows lensing probes structures along the light-path with wavenumber $k_{||} \sim 1/r$, where r is the distance to the source. If we fix the sources at a single distance from the observer, the lensing inversion equation reduces to the 2-D Kaiser-Squires relation

$$\kappa = -\frac{3}{2} \frac{r^2}{\lambda_H^2} \left(\frac{\Omega_m}{a} \right) \left(\frac{k_{\perp}^2 r^2}{1 + k_{\perp}^2 r^2} \right) \delta(k_{||} = r^{-1}, k_{\perp}) \quad (24)$$

and so for large distances or small structure, we find that the lensing signal increases as a function of the source distance as $\kappa \sim r^2$.

C. Measurement of the mass-density field

These relations allow us to estimate the accuracy for reconstructing the 3-D density field from gravitational lensing. The covariance matrix of Fourier modes of the density field, $\delta(\mathbf{k})$, estimated from lensing is

$$C_{\delta\delta}(\mathbf{k}) = P_{\delta}(k) + \frac{4}{9} \left(\frac{ak^2\lambda_H^2\mu^2}{\Omega_m(1-\mu^2)} \right)^2 \frac{e_{\text{rms}}^2}{n}, \quad (25)$$

where we have assumed the underlying density field is statistically homogeneous and isotropic;

$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = (2\pi)^3 P(k) \delta_D(\mathbf{k} - \mathbf{k}'), \quad (26)$$

and $\delta_D(\mathbf{k})$ is the Dirac delta function. The second term arises due to Poisson sampling of the shear field by source galaxies, where $e_{\text{rms}} \approx 0.4$ is the intrinsic dispersion of galaxy ellipticities, and n is the density of sources.

The uncertainty on a measurement of the matter power spectrum from a finite galaxy lensing survey is

$$\frac{\Delta P_{\delta}(k)}{P_{\delta}(k)} = \frac{2\pi}{\sqrt{k^3 d \ln k V_{\text{eff}}}}, \quad (27)$$

where $V_{\text{eff}} = \int d^3r [P(k)/(P(k) + N(\mathbf{k}))]^2$ is the effective volume of the survey, where $N(\mathbf{k})$ is the Poisson noise, and we sample the matter power spectrum in logarithmic intervals of $d \ln k$. Equation (27) shows the fractional uncertainty per modes divided by the square root of the number of effective independent modes that will fit into the survey volume. For a finite survey the angular part of the integral should have limits $1/(kR) \leq \mu \leq \sqrt{1 - 1/(kR)^2}$ and $k \geq \sqrt{2}/R$, where R is the size of the survey. For small scales this is dominated by shot-noise and $k^3 V_{\text{eff}}(k)/2\pi^2 \approx 27/(56\pi)(nPe_{\text{rms}}^2)^2(k\lambda_H)^2(R/\lambda_H)^{10}(\Omega_m/a)^4$, which is strong function of the volume.

Figure 1 shows the 3-D matter power spectrum and the expected uncertainty from an idealised galaxy lensing survey. The dotted line is the linear power spectrum, while the solid line is the nonlinear power [17]. The model is a Λ CDM model, with $\Omega_{\Lambda} = 0.7$ and $\Omega_m = 0.3$, normalised to the present-day cluster abundance [18]. The

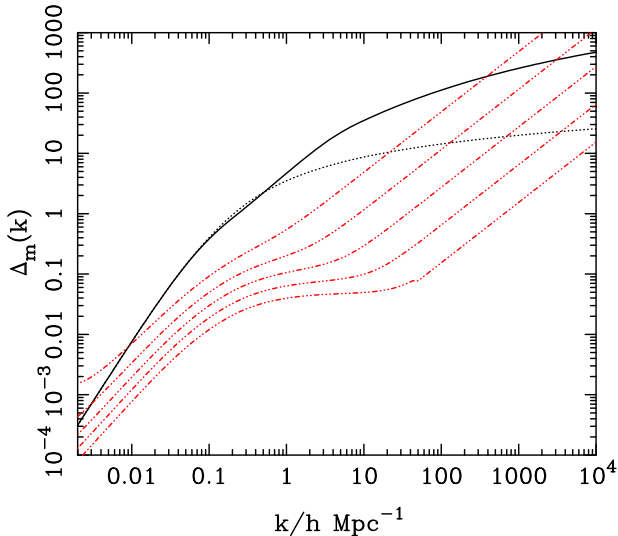


FIG. 1. The dimensionless matter power spectrum, $\Delta_m(k) = \sqrt{k^3 P(k)/2\pi^2}$, for a Λ CDM cosmology, with $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$, normalised to the observed abundance of clusters. The dotted line is linear theory, the solid line is nonlinear theory. The dot-dashed lines are the expected uncertainty on a measurement of the power from a set of large-scale galaxy lensing surveys.

upper dot-dashed line shows the uncertainty for a survey with source density $n = 10^{-3} h^{-1} \text{Mpc}^{-3}$ and volume $V = 10^9 h^{-1} \text{Mpc}^3$, similar to the Sloan Digital Sky Survey. The lower lines are for larger surveys with the lower line for a survey ten times larger. Clearly the 3-D matter power spectrum can be recovered to good accuracy.

III. SUMMARY

I have derived an exact expression for the reconstruction of the full three-dimensional matter density field from measurements of weak gravitational lensing of source galaxies. This can be applied to matter fields in both the linear and nonlinear clustering regime, since the method is based on a new relation between the Newtonian potential and the gravitational lensing potential. The lensing potential can be estimated from observable lens shear or convergence fields. A new sheet-like degeneracy arises due to a constant of integration of the observable fields which is an arbitrary function of distance. I have shown that this function can be removed by averaging over light paths.

I have derived the normal modes of the lens fields and the reconstruction equation. Using a small-angle and plane-wave approximation I have derived a simplified set of equations. The small-angle approximation allows us to estimate the accuracy to which a realistic galaxy lensing survey can reconstruct the full matter distribution. I find that the reconstruction of the full distribution is possible with current surveys, if the data quality is sufficiently good. 3-D imaging is an interesting challenge for future

lensing surveys. Since the addition of depth information adds an important new aspect to lensing, estimation of statistical quantities such as the 3-D matter power spectrum should be possible with near-future surveys. Combined with the distribution of galaxies from the same survey, this will lead to direct constraints on the environmental aspects of galaxy formation, as well as a direct probe of the cosmological distribution of Dark Matter, which in turn will place strong constraints on theories of structure formation and the nature of the Dark Matter.

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